MATH2050C Selected Solution to Assignment 13

Supplementary Problems

1. Show that for $x > 0$, the sequence $\{a_n\}, a_n = (1+x/n)^n$ is strictly increasing and bounded from above by $\sum_{k=0}^{\infty} x^k/k!$.

   Solution By Binomial Theorem, $(1+x/n)^n = \sum_{k=0}^{n} C_k^n (x/n)^k$. The general term is

$$C_k^n \frac{x^k}{n^k} = \frac{n(n-1)\cdots(n-k+1)}{k!}\frac{x^k}{n^k} = \left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k-1}{n}\right)\frac{x^k}{k!} \le \frac{x^k}{k!} .$$

   Hence

$$a_n \le \sum_{k=0}^{n} \frac{x^k}{k!} \le \sum_{k=0}^{\infty} \frac{x^k}{k!} .$$

   $\{a_n\}$ is bounded from above. Next, the k-th term in a_{n+1} is

$$\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)\cdots\left(1-\frac{k-1}{n+1}\right)\frac{x^k}{k!}$$

   which is greater than the k-th term of a_n, hence $a_n < a_{n+1}$.

2. Show that for each $m \ge 1, E(x) \ge \sum_{k=0}^{m} x^k/k!$ and conclude $E(x) = \sum_{k=0}^{\infty} x^k/k!$.

   Solution The k-th term in a_n is equal to

$$\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k-1}{n}\right)\frac{x^k}{k!} ,$$

   which is less than $x^k/k!$. Hence for a fixed m, for all $n \ge m$,

$$\sum_{k=0}^{m}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k-1}{n}\right)\frac{x^k}{k!} \le a_n \le E(x) .$$

   Letting $n \to \infty$, we obtain $\sum_{k=0}^{m} x^k/k! \le E(x)$. Now, letting $m \to \infty, \sum_{k=0}^{\infty} x^k/k! \le E(x)$. From the two-way bound we conclude $E(x) = \sum_{k=0}^{\infty} x^k/k!$.

3. Show that for $x < 0, E(x) = \lim_{n\to\infty} a_n$ exists and $E(x)E(-x) = 1$.

   Solution From the relation $(1+a)(1-a) = 1-a^2$ we have

$$\left(1+\frac{x}{n}\right)^n = \frac{\left(1-\frac{x^2}{n^2}\right)^n}{\left(1-\frac{x}{n}\right)^n} .$$

   Noting that $-x^2/n^2 > -1$ for large n, by Bernoulli's inequality $((1+a)^n \ge 1+na, a > -1)$,

$$\left(1-\frac{x^2}{n^2}\right)^n \ge 1 - n\frac{x^2}{n^2} = 1 - \frac{x^2}{n} .$$

   We have

$$1-\frac{x^2}{n} \le \left(1-\frac{x^2}{n^2}\right)^n \le 1 ,$$

for all large n. By Squeeze Theorem we conclude

$$\lim_{n\to\infty}\left(1-\frac{x^2}{n^2}\right)^n = 1 \ .$$

Applying the Quotient Rule, for $x<0$,

$$E(x)\equiv\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = \frac{\lim_{n\to\infty}\left(1-\frac{x^2}{n^2}\right)^n}{\lim_{n\to\infty}\left(1-\frac{x}{n}\right)^n} = \frac{1}{E(-x)} \ .$$

4. Show that for $x>0, a, b\in\mathbb{R}$, $x^ax^b=x^{a+b}$ and $(xy)^a=x^ay^a$.

Solution First,

$$x^ax^b = E(a\ln x)E(b\ln x) = E(a\ln x + b\ln x) = E((a+b)\ln x) = x^{a+b}.$$

Next,

$$(xy)^a = E(a\ln xy) = E(a\ln x + a\ln y) = E(a\ln x)E(a\ln y) = x^ay^a \ .$$